Computer Simulation and Applications in Life Sciences

Fractals and Simulation of Recursive Growth Processes
Slides

- Based on the chapter 11 (Fractals) of J. Clinton Sprott, Chaos and Time Series Analysis, Oxford (2006)
The Fractals

- Falconer describes fractals as follows
  - They have a fine structure (detail on arbitrarily small scales)
  - They are too irregular to be described by ordinary geometry, both locally and globally
  - They have some degree of self-similarity
  - Their fractal dimension is greater than their topological dimension
  - They often have unusual statistical properties such as zero or infinity average and variance
  - They are defined in a simple way, often recursively
Examples of fractal structures
Why study fractals? (1)

- Dynamic Systems generate a trajectory $x(t), t = 0, \ldots$ in some state space $X$
- These trajectories can look very complicated
- Fractal geometry is required to analyse and describe these dynamics
- By computing the fractal dimension of a time series we can estimate the number of active variables in the system

Example: Trajectory in 2-D space of a random, walk (Brownian motion) with 100000 steps
Why study fractals? (2)

- Growth processes are often recursive and can generate structures with a fractal geometry.
- These structures can also have a fractal geometry.
- Lindenmayer systems (L-systems) are used to characterize these systems.
- Self-similarity is an important aspect in these systems.
Why study fractals? (3)

- The convergence of numerical/iterative computation methods used in simulation depends often critically on starting values.
- The boundary between convergent and divergent starting points is a fractal.
Fractals - Deterministic vs. Stochastic

- Fractals can be \textit{deterministic}, where they can be exactly self similar, e.g. Lindenmayer systems.
- Fractals can be \textit{random} where they are statistically self-similar, e.g. Brownian motion.
Historical notes

- Fractals are studied in mathematics for more than a century
- However, they were long time considered as mathematical curiosities and not related to nature
- Books by Benoît Mandelbrot and Barnsley popularized the study of fractals and showed their relevance for natural science
- The onset of computers allowed to simulate fractal growth processes and enabled new analysis techniques
- Fractal geometry has become an indispensable topic for the study non-linear dynamical systems in systems modeling and simulation
Note on the word ‘fractal’

- The word fractal was coined by Benoit Mandelbrot (1977)
- It translates to ‘irregular’ (lat.: *frangere* break into irregular fragments)
- Also suggests an object with fractional dimension
- It can be used as adjective and noun
Self-similarity

- *Self-similarity* means that small pieces of the object resemble the whole in some way.
- In other words, one can say that these properties are *scale invariant*.
- Remark: In natural fractals self-similarity can be observed typically in no more than 3 scales.

First 3 iterations of the Menger Sponge
Examples of fractals
Cantor dust

- Perhaps the prototypical fractal was studied by Cantor 1883
- Simulation Algorithm for generating the triadic Cantor dust:
  1. Start with a line segment of unit length
  2. Remove the middle third
  3. Take the remaining pieces and remove the middle third
  4. Goto 3

Georg Ferdinand Ludwig Phillip Cantor: Grundlagen einer allgemeinen Mannigfaltigkeitslehre, Mathematische Annalen 21, 545-91
Cantor dust

- The initial object is called initiator
- The object after one step is called the motif
- The objects generated after finitely many applications are called prefrectals
- The self-similarity of the prefrectals is obvious: Each remaining fragment looks like its parental fragment, except being 3 times smaller
- The process can be repeated infinite many times, giving rise to the Cantor dust

G. Cantor: Grundlagen einer allgemeinen Mannigfaltigkeitslehre, Mathematische Annalen 21, 545-91
Cantor dust

- The Cantor dust contains infinitely many objects
- The Cantor dust has some surprising properties:
  - The number of connected subsets in the cantor is uncountable
  - Recall: A set is countable, if and only if its elements can be listed as a sequence of numbers with every element in the set occurs at a specific number (place) in the list.
  - The set is totally disconnected, i.e. each element of the set is separated from its neighbours by a gap
  - Each element has infinite many neighbours within any finite neighborhoods
Measuring the cantor dust (more strange properties …)
- The total length of all elements in the cantor dust together is zero
- You can compute it:
  \[ l_\infty = \lim_{t \to \infty} \left( \frac{2}{3} \right)^t \]
- Thus the Cantor set is more than a finite collection of points, but less than a collection of line segments
- It makes sense, to say that its dimension is inbetween 0 (finite point set) and 1 (line segment set)
- Later we will determine the Hausdorff-Besikovitch dimension of the cantor dust \( C \) to be:
  \[ d_H(C) = \frac{\log(2)}{\log(3)} = 0.6309 \]
Topological dimension (sketch)

- Basic idea (for bounded sets):
  - The dimension of the set F has to determined
  - The dimension of the empty set is -1
  - The dimension of a finite point set point is 0
  - The boundary of a set of dimension N has dimension N-1
  - e.g. Curves (D=1) are the boundary of areas (2-D), surfaces (2-D)
    are the boundary of 3-D volumes, etc …

- For a more precise definition, topologies, closures, and boundaries need to be axiomatically defined.

- A topology is a system of open set, each of which a subsets from some space M, which is closed under intersection and (infinite) union; The empty set and M are both open; Complements of open sets are called closed sets. The boundary of an open set X is the smallest closed set, that contains X, excluding X.
Hausdorff Besikovitch Dimension

- The Hausdorff Besikovitch Dimension is a measure of how fast a set of spheres of radius $\epsilon$ needed to cover a set approaches infinity for a shrinking radius $\epsilon$.
- Given a (fractal) set we can draw a sphere of radius $\epsilon$ around each point of that set.
- Under certain circumstances we can remove spheres, such that the set of spheres still covers the entire set.
- The number of spheres of radius $\epsilon$ minimally needed to cover the set is called $N(\epsilon)$.
- $N(\epsilon)$ grows with $\epsilon$ with a speed $N(\epsilon) \sim 1/ (\epsilon^D)$ for some $D$, where $D$ is called the Hausdorff Besikovitch Dimension:

$$D = \log N(\epsilon) / \log \left( \frac{1}{\epsilon} \right)$$
Cantor dust

- Even more surprising properties:
  - Any real number in the interval $0 < X < 2$ can be represented exactly as a sum of two elements of the cantor dust
  - It consists of all elements in the unit interval, the ternary representation of which contains only 0 and 2
Cantor curtains

- The fraction removed in the middle can be taken differently to zero and one
- The variation of the middle fraction
- A stack of Cantor sets with different middle sections removed gives rise to the Cantor curtains (Mandelbrot 1983)
- The fractal dimension of this stack objects changes locally from 1 at the lower edge towards two at the upper edge

Figure: The cantor sets for gradually increased middle fractions of the motifs displayed as a stack.
The Devil's Staircase

- The devil’s staircase is found by integrating the triadic Cantor set along its extent (Hille and Tamarkin, 1929).
- The integral $D(l)$ indicates which percentage of the cantor dust has been gathered in the part of the unit interval up to a length of $l$.
- The functions contains infinite many small steps (staircase).

Hille and Tamarkin (1929): Remarks on a known example of a monotonic function, American Mathematics Monthly 36, 255-64
The devil’s staircase

- The length of the staircase is 2
- Its fractal dimension is 1 and therefore its ‘status’ as a fractal is debatable
- The devil’s staircase can be observed in many physical systems and heartbeat modeling
Fractal curves

- Like the devil’s staircase, many fractals are described by curves.
- Another way to generate fractals is given by the following generic algorithm:
  1. Start with a line
  2. Do not remove parts of it, but rather bend in a self-similar way
- There are countless variations of this theme …
The Hilbert curve

- The Hilbert curve is an example of a non-intersecting, space-filling curve.
- What is its initiator and motif?
- It converges towards a filled plain.
- Adjacent points on the plane are not always adjacent on the curve, but the opposite holds.
Peano curves

- The Hilbert curve is an example of Peano curves
- These are all curves that never intersect each other, and their iterates converge towards the unit plane

Peano, G. (1890). Sur une courbe, qui remplit une aire plane; Mathematische Annalen 36, 157-60 (translation in Peano 1973)
Von Koch Snowflake

- The von Koch snowflake is formed starting from an even sided triangle (initiator)
- The Motif is
Von Koch Snowflake

- Lemma: The line around the van Koch snowflake fractal has infinite length
- Proof:
  1. The initial line has length 3.
  2. The first iterate has length $3 \times (4/3)$.
  3. In that iterate every line segment has length $1/3$.
  4. In the second iterate each of the $3 \times 4$ line segments is replaced by a line segment of length $4/3 \times 1/3$; In total we get $3 \times 4 \times 4/3 \times 1/3 = 3 \times 4/3 \times 4/3$
  5. Likewise, we show the next iterate has length $3 \times (4/3)^3$; the $t$-th iterate has length $3 \times (4/3)^t$ and

$$\lim_{t \to \infty} 3 \times (4/3)^t = \infty$$
Von Koch Curves

- Variations of the Von Koch Snowflakes are called Van Koch curves
- Some properties of the Von Koch Snowflake and similar curves
- The Von Koch snowflake has finite volume
- The boundary has infinite length
- The curve is nowhere differentiable
- Its fractal dimension is inbetween 1 and 2
Country borders as fractal curve

- Like the Von Koch Curves, also lakes and/or islands and country borders are often best viewed as fractals and their length can be easily underestimated
- The length of the boundary depends on the scale of the measurement
- The ruggedness of shorelines can be compared by their fractal dimension. The statistical dimension of Britain’s shoreline is about 1.2 and that of Norway’s shoreline 1.5.

L.F. Richardson (1961) The problem of contiguity: an appendix on the statistics of deadly quarrels, Yearbook of the society of general systems research 6, 139-87