Justify your answers and supply enough detail to support your answers.

1. What is the definition of a stochastic process (aka random process)?

2. What is the definition of discrete-time dynamical system?

3. What is the definition of periodic orbit of period \(k\) (aka period-\(k\) orbit)?

4. Consider the graph of a function \(f(x)\) on the handout. Write your work for this question on the handout.
   a. Determine graphically the fixed points of the specified function.
   b. Let the initial point be \(x_0 = 0.2\). Determine graphically by using a cobweb the first 3 points of the orbit of \(x_0\). Note that \(x_0\) is the first point of the orbit.
   c. Determine whether the fixed points of the specified function are sinks or sources graphically (i.e., using cobwebs again).
   d. In the handout the second graph shows the function \(f(x)\) and its second iterate \(f^2(x)\). Determine graphically the period-2 points of this function.

5. Compute analytically the basin of attraction of the attracting fixed point of the logistic map \(f(x) = 2x(1 - x)\).

6. Consider the family of logistic maps \(f_a(x) = ax(1 - x)\) where \(0 < a \leq 4\). The domain of each of the maps \(f_a\) is \([0,1]\).
   a. What is the range of each family member \(f_a\)?
   b. Compute for each of the members its fixed points.
   c. State the theorem which characterizes the nature of the fixed points for maps in general (i.e., it tells us in most cases whether the fixed point is repellent (is a source) or attractive (is a sink)). Use this theorem for the special case of members of the logistic family. That is for each member \(f_a\) classify the nature of its fixed points.

7. In the assignments you have seen that the logistic map can exhibit aperiodic orbits for certain parameter values, but how do we know that this is really chaos? To be called "chaotic", a system should also show sensitive dependence on initial conditions, in the sense that neighboring orbits separate exponentially fast, on average. We will now proceed to quantify this sensitive dependence by defining the Liapunov exponent and at
same time give an intuitive motivation for this definition. Given some initial condition \( x_0 \), consider a nearby point \( x_0 + \delta_0 \), where the initial separation is extremely small. Let \( \delta_n \) be the separation after \( n \) iterates. If \( |\delta_n| \approx |\delta_0| \exp^{n\lambda} \), then \( \lambda \) is called the Liapunov exponent. A positive Liapunov exponent is a sign of chaos. Derive a more precise and computationally useful formula for \( \lambda \) (the Liapunov exponent for the orbit starting at \( x_0 \)) by taking logarithms and noting that \( \delta_n = f^n(x_0 + \delta_0) - f^n(x_0) \).

8. a There is close relationship between iterative function systems and ordinary differential equations. Design a model for population growth with \( A \) being the birth rate and \( B \) being the death rate, using (1) a discrete state and iterative function systems, and (2) a continuous state and a differential equation system.

b Given the differential equation model from the previous question, compute numerically the state of the population in \( t = 2002 \) for an initial population size of 500 in \( t = 2000 \), \( A = 3 \), \( B = 2 \), and a stepsize of 1 using the Euler method.

c What is the difference between Runge Kutta and the Euler Method?

9. Lindenmayer Systems and Cellular Automata

a Given an axiom \( \omega = FF \) and production rule \( F \rightarrow F + F + \), and angle \( \rho = 45^\circ \): compute the first two derivations of a Lindenmayer systems and represent them graphically.

b Represent the von Koch Snowflake as an Lindenmayer system by specifying angle, axiom, and production rule.

c Compute the capacity dimension of the von Koch Snowflake.

10. How many rules can be defined for a cellular automaton with a Moore neighborhood of radius 2 and a binary state space?

11. a Give a definition of an absorbing state in a Markov Chain and also the definition of an absorbing Markov Chain.

b For each of the following two transition matrices \( A \) and \( B \) specifying Markov Chains determine whether they determine an absorbing, regular or ergodic Markov Chain.

c Determine the expected number of steps before the Markov Chain specified by the second transition matrix (i.e., matrix \( B \)) in the previous question is absorbed, if the MC starts in state one (\( s_1 \)).

\[
A = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
\frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\
0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]