1 Assignments

1. For the logistic family \( f_a(x) = ax(1-x) \) we showed in class that \( \forall a, 3 < a, f_a(x) \) has period-2 orbit. Complement the slides of September 23 with all the details showing the above statement.

Solution First we get the fixed points of \( f_a \). These will be useful in getting the period-2 points. We solve \( ax(1-x) = x \) for \( x \): The fixed points for \( f_a \) are 0 and \( \frac{-1+a}{a} \).

(The following is just stated for completeness sake. The derivative of \( f_a \) is equal \( f_a'(x) = a - 2ax \). Hence, \( |f_a'(0)| = a \) and \( |f_a'(\frac{-1+a}{a})| = |2-a| \). For members with \( a < 1 \), 0 is a sink. For members with \( a > 1 \), 0 is a source. (For \( a = 1 \) we have a sink, if the domain of \( f_1 \) is restricted to \([0,1]\).) For members of the logistic family with \( 1 < a < 3 \), we have that \( |f_0'(\frac{-1+a}{a})| = |2-a| < 1 \), so we have sink. For \( a < 1 < a > 3 \) the derivative is bigger than 1 in absolute value, so we have a source. End of Aside.)

The period-two points of \( f_a \) are contained in the set of fixed points of \( f_a^2 \). We compute the defining expression of \( f_a(f_a(x)) \). It is equal to \( a^2(1-x)x(1-a(1-x)x) \). The relevant equation for the fixed points is \( f_a(f_a(x)) = x \) or \(-x + a^2x + (-a^2 - a^3)x^2 + 2a^3x^3 - a^3x^4 = 0 \). One solution of this is \( x = 0 \). We will factor out \( -1 + a^2 + (-a^2 - a^3)x + 2a^3x^2 - a^3x^3 = 0 \).

Another solution of this or previous equation is \( \frac{-1+a}{a} \). So we will factor out \(-\frac{-1+a}{a} + x: \frac{-1+a^2+(-a^2-a^3)x+2a^3x^2-a^3x^3}{-\frac{-1+a}{a} + x} = -a - a^2 + a^2x + a^3x - a^3x^2 \).

Solving the quadratic equation \(-a - a^2 + a^2x + a^3x - a^3x^2 = 0 \) gives as solutions \( \frac{(-1-a)a - a\sqrt{-3 - 2a + a^2}}{2a^2} \) and \( \frac{(-1-a)a - a\sqrt{-3 - 2a + a^2}}{2a^2} \), or slightly simplified: \( \frac{(1+a)\sqrt{-3 - 2a + a^2}}{2a^2} \) and \( \frac{(1+a)\sqrt{-3 - 2a + a^2}}{2a^2} \). There will be two real solutions if \(-3 - 2a + a^2 > 0 \) or equivalently: \( a^2 - 2a + 1 - 4 > 0 \) or equivalently \((a-1)^2 > 4 \) or equivalently \( a - 1 < -2 \) or \( a - 1 > 2 \) or equivalently \( a < -1 \) or \( a > 3 \). Since the parameter \( a \geq 0 \) we deduce that \( a > 3 \) (i.e., for each \( a > 3 \), \( f_a(x) = ax(1-x) \) has a period-two orbit.

Number of credit points is 4. Refer any questions you might have to deutz@liacs.nl.
2. In class we showed that the period-2 orbits are stable (that is, they are sinks) for the maps $f_a(x) = ax(1 - x)$, $\forall a, 3 < a < (1 + \sqrt{6}) \approx 3.449$. Write up the details of this proof.

**Solution**

Recall that for a quadratic equation $Ax^2 + Bx + C = 0$ the sum of the roots is equal to $-B/A$ and the product of the roots is $C/A$.

Let $p_1, p_2$ be the period-two orbit of the map $f(x) = ax(1 - x)$. We obtained these as the solutions of the quadratic equation $\frac{a}{a^2 + a^3 x^2} = \frac{1 + a}{a^2}$ and the product is $\frac{a + a^2}{a^2}$ (take $A = a^3, B = -(a^2 + a^3)$ and $C = a + a^2$).

In order to determine the stability we need to compute $|f'(p_1) * f'(p_2)|$.

This is equal to:

\[
|a(1 - p_1)a(1 - p_2)| = \\
|a^2(1 - 2(p_1 + p_2) + 4p_1p_2)| = \\
|a^2(1 - 2 \frac{1 + a}{a} + 4 \frac{1 + a}{a^2})| = \\
|a^2 - 2a(1 + a) + 4(1 + a)| = \\
|-(a^2 + 2a + 4)| = \\
|-(a^2 - 2a - 4)| = \\
|(a^2 - 2a + 1 - 5| = \\
|(a - 1)^2 - 5|
\]

The period-two orbit will be a sink, if

\[
|f'(p_1) * f'(p_2)| = |(a - 1)^2 - 5| < 1
\]

The latter is equivalent to:

\[
-1 < (a - 1)^2 - 5 < 1
\]

or

\[
4 < (a - 1)^2 < 6
\]

or

\[
3 < a < \sqrt{6} + 1
\]

3. Periodic Windows and the Logistic Family. It is very intriguing to discover in the bifurcation diagram periodic windows for $a > a_\infty$. The period-three windows that occurs near $3.8284 \cdots \leq a \leq 3.8415 \cdots$ is the most conspicuous one. Suddenly, against a backdrop of chaos, a stable (that is, sink) period-3 orbit appears out of the blue. The question is to understand how this period-3 orbit is created. (The same mechanism accounts for the creation of all the other windows, so it suffices to consider the simplest case.) Study this problem by looking at a gliding family of maps $f_a(f_a(f_a(x)))$ where $a$ varies from say $3.8415 \cdots$ down to $3.8284 \cdots$. (Use gnuplot or octave or matlab or C++ etc to plot $f_a^3$ for the various values of $a$.)

For a solution see lecture notes.
4. Plot the bifurcation graph of the map \( x_{n+1} = a \sin(\pi x_n) \) for \( 0 \leq a \leq 1 \) and \( 0 \leq x \leq 1 \). What do you notice? Might this be true more generally?

5. Define what we mean by bifurcation. What kind of bifurcation occurs at \( a = 1 \) and \( a = 3 \) for the logistic family.

6. Define what we mean by a chaotic system.