1 Assignments

1. Describe at least one question you posed yourself in your study of the logistic family \( f(x) = ax(1-x) \) and describe how you went about to get the answer.

2. Consider the map \( f(x) = x^3 + x \). Domain=Range=\( \mathbb{R} \). Determine the fixed points of this map. Decide whether they are sinks or sources. You will have to work without the Theorem of the 9-September lecture stated below, since it does not apply.

**THM 1** Let \( f \) be a differentiable map on \( \mathbb{R} \), and assume that \( p \) is a fixed point of \( f \). Then:

- a If \( |f'(p)| < 1 \), then \( p \) is a sink.
- b If \( |f'(p)| > 1 \), then \( p \) is a source.

**Solution:** Consider any positive number \( p \). Clearly \( f(p) \) is at a greater distance from the origin (the fixed point) than \( p \) itself. For \( f(p) = p^3 + p \), and therefore it is further away from the origin by a positive amount of \( p^3 \). Similarly for any negative number \( n \), \( f(n) = n^3 + n \) is at a greater distance from the origin as compared to \( n \). (NB Both \( n^3 \) and \( n \) are negative, thus from \( n \) which is on the left of the origin you move further in the direction of \(-\infty\) by an amount of \( |n^3| \). These remarks make it clear that the origin is a source (repellor).

NB.: \( g(x) = x - x^3 \) has a sink at 0, again see cobweb; 0 is repelling from one side and attracting from the other for map \( h(x) = x + x^2 \).

3. Is the period-two orbit of the map \( f(x) = 2x^2 - 5x \) on \( \mathbb{R} \) a sink, a source, or neither. Use the Theorem of the 16-September lecture for the classification.

**Solution:** As a prelim step we compute the fixed points of \( f \) by solving
Consider the map. Find the period-2 orbit and determine whether it is a sink, source or a repelling orbit by using the theorem of the lecture. We need to compute \( f'((1 + \sqrt{2}), f'(1 + \sqrt{2})). \) As \( f'(x) = 4x - 5 \) we need to compute \((4(1 + \sqrt{2}) - 5)(4(1 - \sqrt{2}) - 5)\) which is equal to \(| -31 |. \) Since this is bigger than 1, we have a source period-two orbit or a repelling period-two orbit.

4. Consider the map \( f(x) = 4x(1 - x). \) Domain=Range=[0,1]. The function has as domain and range [0,1]. The maximum the function attains on this interval is 1 and the minimum is 0. So \( f([0,1]) = [0,1]. \)

\( f(x) = x \) or equivalently \( 2x^2 - 5x = x. \) Solving this equation we get 0 and 3 as fixed points of \( f. \)

The period-two points are found among the fixed points of the map \( f^2. \) (Clearly, the fixed points of \( f \) are also fixed points of \( f^2. \) The fixed points of \( f \) are definitely not period-two points; if you want you can call them period-one points. So the set of fixed points of \( f^2 \) contains more points than period-two points.)

\[
f^2(x) = f(f(x)) = 2(2x^2 - 5x^2 - 5(2x^2 - 5x) = 8x^4 - 40x^3 + 40x^2 + 25x;
\]

the fixed points of \( f^2 \) we get by solving the equation: \( f^2(x) = x \) or equivalently: \( 8x^4 - 40x^3 + 40x^2 + 25x = x \) (two solutions of this equation are 0 and 3 being fixed points of \( f \).) After simplification we get: \( 2x^3 - 10x^2 + 10x + 6 = 0 \) and \( x = 0. \) Since, 3 is also a solution we know that we can write \( 2x^3 - 10x^2 + 10x + 6 \) as a product of \( x - 3 \) and some second degree polynomial, namely: \( 2x^3 - 10x^2 + 10x + 6 = (x - 3)(2x^2 - 4x + 2). \) So in order to solve \( 2x^3 - 10x^2 + 10x + 6 = 0, \) we focus on solving \((x - 3)(2x^2 - 4x + 2) = 0. \) This reduces to solving \( x - 3 = 0 \) and \((2x^2 - 4x + 2) = 0. \) Solutions of the latter are \( 1 \pm \sqrt{2}. \)

So \( \{1 + \sqrt{2}, 1 - \sqrt{2}\} \) is a two orbit of \( f. \) Next we determine whether this is an attracting or repelling orbit by using the theorem of the lecture: we need to compute \( |f'(1 - \sqrt{2})| \cdot f'(1 + \sqrt{2})|. \) As \( f'(x) = 4x - 5 \) we need to compute \(|(4(1 + \sqrt{2}) - 5)(4(1 - \sqrt{2}) - 5)|\) which is equal to \(| -31 |. \) Since this is bigger than 1, we have a source period-two orbit or a repelling period-two orbit.

\[
\text{Solution} \quad \text{We first convince ourselves of the existence of a period-two orbit geometrically. The graph of } f^2 \text{ is shown in Figure 6. It is not hard to verify by hand the general shape of the graph. First, note that the image of } [0,1] \text{ under } f \text{ is } [0,1], \text{ so the graph is entirely within the unit square. Second note that } f(1/2) = 1 \text{ and } f(1) = 0, \text{ hence } f^2(1/2) = 0. \text{ Further, since } f(a_1) = 1/2 \text{ for some } a_1 \text{ between 0 and } 1/2 \text{ (for } f([0,1/2]) = [0,1] \text{ and the intermediate value theorem for continuous functions), it follows that } f^2(a_1) = 1. \text{ Similarly, there is another number } a_2 \text{ such that } f^2(a_2) = 1. \text{ It is clear from Figure 6 that } f^2 \text{ has 4 fixed points; they either have order 1 or order 2. Two of them are known to us: they are the fixed points of } f. \text{ The new pair of points (let us call them } p_1 \text{ and } p_2 \text{) make up a period-two orbit: that is } f(p_1) = p_2 \text{ and } f(p_2) = p_1. \text{ This argument convinces us that there is a period-two orbit. Below we will compute the points } p_1 \text{ and } p_2 \text{ explicitly.}
\]

Now we will compute the period-two orbit of \( f \) explicitly.

First the fixed points of \( f \) are computed as follows:

\[
4x(1 - x) = x \quad (1)
\]
\[4(1 - x) = 1 \text{ or } x = 0 \quad (2)\]
\[1 - x = 1/4 \text{ or } x = 0 \quad (3)\]
\[1 - x = 1/4 \text{ or } x = 0 \quad (4)\]
\[x = 3/4 \text{ or } x = 0 \quad (5)\]

For completeness sake we determine the nature of the fixed points of \(f\). The derivative is \(f'(x) = 4 - 8x\). By computing \(f'(0) = 4\) and \(f'(3/4) = -2\). Thus both fixed points are unstable (repellent. We have a map without sinks!

The period-two points are computed from the equation \(f(f(x)) = x\) (that is, they are fixed points of \(f^2\):

\[f^2(x) = f(f(x)) \quad (6)\]
\[f(f(x)) = 4f(x) - 4(f(x))^2 \quad (7)\]
\[4f(x) - 4(f(x))^2 = 4(4x - 4x^2) - 4(4x - 4x^2)^2 \quad (8)\]
\[4(4x - 4x^2) - 4(4x - 4x^2)^2 = -16(1 - 2x)^2(-1 + x)x \quad (9)\]
\[-16(1 - 2x)^2(-1 + x)x = 16(1 - 2x)^2(1 - x)x \quad (10)\]

Next we solve the equation:

\[f(f(x)) = x \quad (11)\]

or equivalently because of the above:

\[16(1 - 2x)^2(1 - x)x = x \quad (12)\]
\[16(1 - 2x)^2(1 - x) = 1 \text{ or } x = 0 \quad (13)\]

So we need to solve the equation \(16(1 - 2x)^2(1 - x) = 1\). We know already one solution of this equation: namely the second fixed point of \(f(x)\), \(3/4\). We will now proceed to solve this equation with the knowledge that \(3/4\) is one of the solutions:

\[16(1 - 2x)^2(1 - x) = 1 \quad (14)\]
\[16(1 - 2x)^2(1 - x) - 1 = 0 \quad (15)\]
\[16(1 - 2x)^2(1 - x) - 1 = 15 - 80x + 128x^2 - 64x^3 = 0 \quad (16)\]

Since \(3/4\) is a root of \(15 - 80x + 128x^2 - 64x^3 = 0\), the expression \(15 - 80x + 128x^2 - 64x^3\) is divisible by \(x - (3/4)\). The factoring can be done by long division or another method would be by comparing coefficients – the latter was not discussed in class. By using long division we get:

\[15 - 80x + 128x^2 - 64x^3 = (x - (3/4)) \ast (5 - 20x + 16x^2) \ast (-4) \quad (17)\]

So instead of solving

\[15 - 80x + 128x^2 - 64x^3 = 0 \]
we can solve

\[(x - (3/4)) \ast (5 - 20x + 16x^2) \ast (-4) = 0\]

This is equivalent to \((x - (3/4)) = 0\) or \(5 - 20x + 16x^2 = 0\). The roots of

\[5 - 20x + 16x^2 = 0\]

are

\[\frac{1}{8}(5 - \sqrt{5})\]

and

\[\frac{1}{8}(5 + \sqrt{5})\]

Hence the period-two orbit is

\[\left\{\frac{1}{8}(5 - \sqrt{5}), \frac{1}{8}(5 + \sqrt{5})\right\}\]

Next we determine the nature of this period-two orbit: \(f'(x) = 4 - 8x\) and \(|f'(x)| = | - 4| > 1\); therefore this period-two orbit is repellent (unstable).

b) Prove that for each positive integer k, there is an orbit of period k.

In the Fig. 3 the graph of the map \(f\) and its second and third iterate are drawn.

**Solution**

We already know that \(f\) has a period-two orbit. Let us start with a simpler question: Does \(f\) have a period-three orbit? Consider \(f([0, a_1]) \ (a_1 \text{ as before })\). This set contains 0 and 1/2 = \(f(a_1)\). Hence, by the intermediate value theorem for continuous functions, we have \([0, 1/2] \subseteq f([0, a_1])\) (as a matter of fact \([0, 1/2] = f([0, a_1])\)). Since \(0 < a_1 < 1/2\), we also have that \(a_1\) belongs \(f([0, a_1])\). Therefore there is a point \(b_1\) between 0 and \(a_1\) such that \(f(b_1) = a_1\). Moreover we can find three other points \(b_2, b_3, b_4\) in \([0, 1]\) with the property that \(f^3(b_2) = f^3(b_3) = f^3(b_4) = 1\); so \(f^3\) has four maxima at height 1 in \([0, 1]\). Since \(f(1) = 0\), we get that \(f^3\) has roots at 0, \(a_1, 1/2, a_2,\) and 1. The graph of \(f^3\) is shown in Figure 7. The above analysis shows that \(f^3\) has 8 fixed points.

The map \(f^3\) has 8 fixed points, two of which were known to be the fixed points of \(f\), namely 0 and 3/4. The period-two points are not fixed points of \(f^3\) – can you show this? There remain six points to account for, which must form orbit-three points.

You should be able to prove to yourself in a similar way that \(f^4\) has 2 fixed points, all in \([0, 1]\). With each successive iteration of \(f\), the number of fixed points of the iterate is doubled. In general, we see that \(f^k\) has \(2^k\) fixed points, all in \([0, 1]\). Of course, for \(k > 1\), \(f\) has fewer than \(2^k\) points of period \(k\). (Recall that the definition of of period-k for the point \(p\) is that \(k\) is smallest number positive integer for which \(f^k(p) = p\).) For example, \(x = 0\) is a period-one point of \(f\) and therefore not a period-k point for \(k > 1\), although it is not a one of the \(2^k\) fixed points of \(f^k\).
We now proceed to show more accurately that for any \( k \), the number of fixed points of \( f^k \) is \( 2^k \). From this fact it will be easy to derive the statement that for any \( k \), there is a period-\( k \) orbit. We know that for \( k = 2 \) the graph of \( f^2 \) attains at two points of \([0,1]\) the maximum value of 1; say at \( r_1, r_2 \). The valleys (minimum=0) occur at 0, 1/2, and 1. The point \( r_1 \) lies between 0 and 1/2, the point \( r_2 \) lies between 1/2 and 1. It is clear that the diagonal \((x = y)\) intersects the two “mountains” in \( 2^2 = 4 \) points. This is our base case for induction on \( k \). The induction hypothesis: We assume that for any \( k \leq N \) (with \( 2 \leq N \)) the number of points in \([0,1]\) in which the function \( f^k \) attains the maximum (equal to 1) is \( 2^{k-1} \) (in other words we have \( 2^{k-1} \) “mountains”, see Figure 1. These \( k \) mountains intersect the diagonal in \( 2^k \) points (i.e., we have \( 2^k \) fixed points).

Figure 1: schematic graph of \( f^k \)

With the induction hypothesis in our hands we show that the statement is true for \( N+1 \). For any \( k \leq N \), we have that there are \( 2^{k-1} \) mountains (and therefore \( 2^k \) fixed points).

Let \( k \) in Figures 1 and 2 be equal to \( N \). Keep these two figures in mind for the following. The points at which the maximum occurs are labeled \( c_1, \ldots, c_{2^k-1} \). In order to show that \( f^{N+1} \) has \( 2^N \) mountains, we intersect the line \( y = c_1 \) with first mountain, this gives rise to two intersection points, the x-coordinates of these intersection points are called \( d_1 \) and \( d_2 \). By construction of \( d_1 \) and \( d_2 \) we know, \( f^N(d_1) = f^N(d_2) = c_1 \). Hence, \( f^{N+1}(d_1) = f^{N+1}(d_2) = 1 \). We also intersect the second mountain with the line \( y = c_2 \), in which case we get two intersection points, the x-coordinates of these points are called \( d_3 \) and \( d_4 \); the N-th iterate of \( f \) in these points is equal to \( c_2 \), and hence \( f^{N+1}(d_3) = f^{N+1}(d_4) = 1 \). With \( y = c_3 \) we intersect the third
Figure 2: schematic graph of $f^k$
6. Consider the bifurcation diagram (see Fig. 4) we discussed in class. The parameter $a$ is bigger than 0 and less than 4. For each member of the logistic family $ax(1-x)$, domain=range=$[0,1]$. Write a program which will generate this diagram. As we discussed in class the diagram can be generated as follows: for each parameter value $a$, choose a (pseudo-)random number as the initial value. Iterate this random initial value for a large number of times. Delete say the first one hundred iterates. The remaining iterates are plotted. Increment the parameter value $a$ by a fixed, chosen step size. For the incremented parameter value you repeat the above: a random initial value is chosen, a large number of iterates is computed, the first one hundred iterates are discarded, the remaining are plotted. Etc. etc.

Solution
See lecture slides for the C++ program.

2 Assignments suggested by the lecture of September 16

1. In the lecture we discussed and proved the following theorem. Let $P(x)$ be a polynomial (examples of polynomials are: $-x^4 + \sqrt{2}x^2 - 5x + 3$, or $x^{1000} + 3.1x^3 - 10$. If $r$ is a root of $P$ (that is, $P(r) = 0$, then $P(x) = (x - r) \cdot Q(x)$ for some polynomial $Q(x)$. The degree of $Q(x)$ will be necessarily lower than the degree of $P(x)$. The latter comes in handy in solving polynomial equations.

   a Write up the proof of this theorem.

   b Show how it comes in handy in the business of solving polynomial equations.
Figure 3: The graphs for $f(x) = 4x(1 - x)$, and $f^2, f^3$

Figure 4: Bifurcation Diagram for the Logistic Family $f(x) = ax(1 - x)$
Figure 5: The graph for \( f(x) = 4x(1 - x) \) First Iterate

Figure 6: The graph for \( f^2 \) Second Iterate of \( f(x) = 4x(1 - x) \)
Figure 7: The graph for $f^3$ Third Iterate of $f(x) = 4x(1-x)$

Figure 8: The graph for $f^4$ Fourth Iterate of $f(x) = 4x(1-x)$